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GEOMETRIC ASPECTS OF EINSTEIN'S THEORY.

BY JAMES PIERPONT.

1. **Historical introduction.** Einstein's General Theory of Relativity marks an epoch in physics only comparable with the Principia of Newton. One of its extraordinary features is its intimate interlacement with the foundations of geometry. In the past geometers have imagined different non-euclidean geometries, while the geometry of physicists has remained euclidean. Einstein has broken with this tradition and has shown how the presence of gravitating matter and electricity may determine the character of circumambient space. We wish to show briefly how this has been effected.

To do this we must devote a few words to the origin of his theory in order that the reader may realize how natural, how almost necessary, his generalized theory is. For a long time physicists have tried to develop a satisfactory theory of electro-magnetic phenomena (e.g., light) in moving media. Let us suppose two persons A , A_1 observe a certain phenomenon and that A_1 moves relative to A with a uniform velocity v . A uses a rectangular system S of coördinates x, y, z and a clock to mark the time t . A_1 uses another rectangular system $S_1(x_1, y_1, z_1)$ and a clock time t_1 , having the same rate as A 's when $v = 0$. Each clock and system of coördinates is at rest relative to its observer. Suppose now that each observer writes down the equations which give an account of the phenomenon. Lorentz showed that a satisfactory theory was obtained if we suppose the equations of A are related to those of A_1 by a certain group of transformations. For simplicity, suppose at a certain instant the axes coincided and that the motion of A_1 is parallel to the x axis. Then these transformations are

$$(1) \quad x = k(x_1 + vt_1), \quad y = y_1, \quad z = z_1, \quad t = k\left(t_1 + \frac{vx_1}{c^2}\right),$$

where c is the velocity of light in vacuo and $k^2(c^2 - v^2) = c^2$.

A fundamental hypothesis of this theory is that the velocity of light is the same for both observers. Suppose at the time t a light signal has reached the point $P(x, y, z)$, and at the time $t + dt$ its coördinates have changed by dx, dy, dz . Then

$$(2) \quad c^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

If P has the coördinates x_1, y_1, z_1 in the system S_1 and dt_1 is the interval of time measured on A_1 's clock corresponding to dt , then the velocity being the same,

$$(3) \quad c^2 = \left(\frac{dx_1}{dt_1} \right)^2 + \left(\frac{dy_1}{dt_1} \right)^2 + \left(\frac{dz_1}{dt_1} \right)^2.$$

From (2) and (3) we have

$$(4) \quad c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0,$$

$$(5) \quad c^2 dt_1^2 - dx_1^2 - dy_1^2 - dz_1^2 = 0.$$

According to the general theory (4) must go over into (5) on applying the transformations (1). This is indeed so.

The next important step we wish to mention in the history of Einstein's theory was taken by Poincare and Minkowski. They interpreted the quadruple (x, y, z, t) as a point in 4-way space whose metric ds (element of arc) is given by

$$(6) \quad ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

If we set $ds = 0$, we get (4). It is at this point that quadratic differential forms make their modest entrance on the scene where later they are to play a dominant rôle.

As we have seen, the form (6) remains unaltered for the transformations (1). But this quadratic form remains unaltered by a much wider group. In fact, if we set $c^2 t^2 = -w^2$, it goes over, aside from the sign, into

$$dx^2 + dy^2 + dz^2 + dw^2$$

which remains unchanged for all rotations of the (x, y, z, w) axes, i.e., for a group of linear orthogonal transformations. Minkowski, therefore, required that the equations of mathematical physics shall remain unaltered for these transformations, and it became incumbent on the advocates of this theory to find such invariant equations. The execution of this program was practically completed by 1910–11; it finds its best exposition in the book of M. v. Laue, "Das Relativitätsprinzip" (first edition, 1911).

The most salient feature of this theory of relativity is the fact that the equations of transformation involve the time t as well as the space coördinates x, y, z . No one had ever ventured to make so revolutionary a step. That it is possible and often desirable to give the equation of dynamics an invariant form was shown by Lagrange a century and a half ago. We refer to Lagrange's classic equations, e.g.,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0,$$

and to the invariant equation of Hamilton,

$$\delta \int L dt = 0.$$

We refer also to the researches of Lamé (e.g., *Leçons sur les Coordonnées Curvilignes*, 1859), to those of Beltrami, and, finally, to Chapters V and VI, "Applications mécanique" and "physiques," in the memorable paper of Ricci and Levi-Civita, "Méthodes de calcul différentiel absolu," in the *Mathematische Annalen*, vol. 54 (1901).

The foregoing theory depends on the hypothesis that the two observers are moving uniformly relative to each other. Since uniform motion is only an exceptional case, one might urge that a theory which depends on such a limitation must be defective and not worthy of much confidence. Drude voices this opinion in his "Optik" (1912), p. 470, where he says "Allein hieraus ist zu erkennen dass diese 'Theorie' keine physikalische Bedeutung haben kann" and scornfully speaks of it as "dieses Zerrbild."

To turn such objections Einstein sought and found (1913-14) a far broader theory which he and others have developed and which is called the *general* theory of relativity. The older theory outlined above is called the *restricted* theory of relativity.

The new theory may be briefly characterized as follows. When the observer A_1 is moving in a general manner, the relation between the two sets of variables x, y, z, t and x_1, y_1, z_1, t_1 is no longer *linear*. Einstein therefore replaces the quadratic form

$$(7) \quad ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

by the general quadratic form

$$(8) \quad ds^2 = \sum_{i,j} a_{ij} dx_i dx_j, \quad i, j = 1, 2, 3, 4, \quad a_{ij} = a_{ji}.$$

To express the equations of physics Einstein has recourse to the *calcul* of Ricci and Levi-Civita mentioned above. The quadratic form (8) is fundamental. From a purely abstract standpoint it furnishes the analytical means of writing down invariant (tensor) equations. On the other hand, by regarding x_1, x_2, x_3, x_4 as coördinates (which in general are not rectangular) of a point in 4-way space, (8) may be regarded as defining the element of arc in this space, i.e., it defines the *metric* in this space since all the metrical properties in the last analysis depend upon (8). The coefficients a_{ij} are only 10 in number since $a_{ij} = a_{ji}$; they are functions of the $x_1, \dots, x_4$. Their determination in any given case depends on the disposition of the gravitating matter and electricity which enter the problem.

For example, in the celebrated problem of the motion of Mercury's perihelion, electrical forces are ignored, the gravitational field is produced by the sun alone, the mass of the planet being neglected in comparison with the sun's. On account of the symmetry of the field it is found that the metric of the surrounding space is given by

$$(9) \quad ds^2 = - (1 - \mu/r)^{-1} dr^2 - r^2 d\varphi^2 - r^2 \cos^2 \varphi d\theta^2 + (1 - \mu/r) dx_4.$$

Here r, φ, θ are polar coördinates, x_4 is the time coördinate, $\mu = 2km/c = 3 \cdot 10^5$ in c.g.s. units, m = mass of sun, $c = 3 \cdot 10^{10}$ the velocity of light, and $k = 6, 7 \cdot 10^{-8}$ is the constant of gravitation.

For $x_4 = \text{constant}$, $dx_4 = 0$ and (9) reduces to

$$(10) \quad - ds^2 = (1 - \mu/r) dr^2 + r^2 d\varphi^2 + r^2 \cos^2 \varphi d\theta^2.$$

This defines the metric of the three-dimensional space around the sun. It is not euclidean.

2. n -way space. Non-euclidean geometry. These terms are full of mystery to the layman, and it must be confessed that, before the advent of Einstein's theory, few mathematicians and still fewer physicists had more than a bowing acquaintance with these subjects. This is partly due to the unfortunate, one might almost say repulsive, way they have often been presented. To begin with the reader should disabuse himself of the idea that there is an n -way space ($n > 3$) in any such way as we think of our 3-way space. For the purpose of this paper it will be helpful to bear in mind that our geometrical terms are merely geometrical names applied to certain analytical expressions or complexes which have their analogues in our ordinary space. We leave it to the metaphysician to decide whether space is one or many, three or n -dimensional, finite or infinite, etc.

Let $x_1, \dots, x_n$ be n variables, the complex $(x_1, \dots, x_n) = x$ we call a *point* and $x_1, \dots, x_n$ its *coördinates*. The totality of the x 's as the coördinates vary form an n -way space R_n . Let p be a variable parameter; if the coördinates $x_1, \dots, x_n$ are related by

$$(11) \quad x_1 = \varphi_1(p), \dots, x_n = \varphi_n(p),$$

the totality or locus of the points x when p ranges over a certain interval is a *curve*. Let p, q be two variable parameters; we say

$$(12) \quad x_1 = \psi_1(p, q), \dots, x_n = \psi_n(p, q)$$

define a *surface*. A relation $F(x_1, \dots, x_n) = 0$ defines a *hypersurface*. Thus $a_1 x_1 + \dots + a_n x_n = 0$ is a hyperplane.

The metric properties of our space R_n depend on our definition of distance. We say the *distance* between the point x and $x + dx$ is ds where

$$(13) \quad ds^2 = \sum a_{ij} dx_i dx_j, \quad i, j = 1, \dots, n, \quad a_{ij} = a_{ji}.$$

In general the a 's are functions of the $x_1, \dots, x_n$.

Example 1. In our ordinary space R_3 (rectangular coördinates)

$$(14) \quad ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

In polar coordinates

$$(15) \quad ds^2 = dx_1^2 + x_1^2 dx_2^2 + x_1^2 \cos^2 x_2 dx_3^2.$$

Example 2. If R_2 is the surface of a sphere of radius r and x_1, x_2 are the ordinary polar coördinates,

$$(16) \quad ds^2 = r^2 dx_1^2 + r^2 \cos^2 x_1 dx_2^2.$$

Example 3. In the restricted theory of relativity

$$(17) \quad ds^2 = c^2 dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2.$$

We call

$$(18) \quad a = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

the *determinant of the form* (13). For the form (15), for example,

$$a = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x_1^2 & 0 \\ 0 & 0 & x_1^2 \cos^2 x_2 \end{vmatrix} = x_1^4 \cos^2 x_2.$$

Associated with the n^2 coefficients a_{ij} are the quantities

$$(19) \quad a^{ij} = \frac{A_{ij}}{a},$$

where A_{ij} is the minor of a_{ij} with its proper sign. Since $a_{ij} = a_{ji}$, we have also $a^{ij} = a^{ji}$. A relation of constant use is

$$(20) \quad \begin{aligned} \sum_{\mu} a_{\lambda\mu} a^{\mu\nu} &= 1 & \text{if } \lambda &= \nu. \\ &= 0 & \text{if } \lambda &\neq \nu, \end{aligned}$$

In fact the well-known relation

$$a_{\lambda 1} A_{\lambda 1} + a_{\lambda 2} A_{\lambda 2} + \dots + a_{\lambda n} A_{\lambda n} = a,$$

on dividing by a , gives the first, and

$$a_{\lambda 1} A_{\nu 1} + \dots + a_{\lambda n} A_{\nu n} = 0$$

gives the other.

Example 4. For the form (15) we have $a^{ij} = 0$ if $i \neq j$, and

$$a^{11} = \frac{x_1^2 \cdot x_1^2 \cos^2 x_2}{a} = 1, \quad a^{22} = \frac{x_1^2 \cos^2 x_2}{a} = \frac{1}{x_1^2}, \quad a^{33} = \frac{1}{x_1^2 \cos^2 x_2}.$$

Another metric notion of great importance is the *angle* θ between two lines, or in general between two curves meeting at a point x . We define this by

$$(21) \quad \cos \theta = a_{ij} \frac{dx_i}{ds} \frac{\delta x_j}{\delta s},$$

where ds , δs are the elements of arc along the two curves and dx_i , δx_j are the coördinate differences of the extremities of these arcs.

Example 5. Using the form (14),

$$(21a) \quad \begin{aligned} \cos \theta &= \frac{dx_1}{ds} \frac{\delta x_1}{\delta s} + \frac{dx_2}{ds} \frac{\delta x_2}{\delta s} + \frac{dx_3}{ds} \frac{\delta x_3}{\delta s} \\ &= l\lambda + m\mu + n\nu, \end{aligned}$$

where l , m , n and λ , μ , ν are the direction cosines of the two curves at the point x .

When $\cos \theta = 0$, we say the curves *meet at right angles* or *orthogonally*.

When p varies from $p = \alpha$ to $p = \beta$, $\alpha < \beta$, the *length of the arc* on the curve (11) is defined to be

$$(22) \quad s = \int_{\alpha}^{\beta} \sqrt{\sum a_{ij} \frac{dx_i}{dp} \frac{dx_j}{dp}} dp.$$

We need one other metric notion, that of area for an R_2 and of volume for an R_n , $n > 2$. Calling this V whether $n = 2$ or $n > 2$, we define

$$(23) \quad V = \int \sqrt{|a|} dx_1 \cdots dx_n.$$

Example 6. Using the metric of example 2 we have $\sqrt{a} = r^2 \cos x_1$, hence for the whole sphere

$$V = \int_0^{2\pi} dx_2 \int_{-(\pi/2)}^{\pi/2} r^2 \cos x_1 dx_1 = 4\pi r^2.$$

It is important to note that the expressions (21), (22) defining angle and volume are invariant under any transformation. To illustrate what this means, suppose we transform the variables $x_1, \dots, x_n$ to $u_1, \dots, u_p$ whereby ds^2 as given by (13) goes over into

$$d\sigma^2 = \sum_{\alpha, \beta} b_{\alpha\beta} du_{\alpha} du_{\beta}, \quad \alpha, \beta = 1, 2, \dots, p.$$

If we make this transformation in (21), we find it goes over into

$$(24) \quad \cos \theta = \sum b_{\alpha\beta} \frac{du_{\alpha}}{d\sigma} \frac{\delta u_{\beta}}{\delta \sigma},$$

i.e., (24) is the same function of the new letters as (21) is of the old. If, in particular, $d\sigma^2 = du_1^2 + du_2^2 + \dots + du_p^2$,

$$\cos \theta = \frac{du_1}{d\sigma} \frac{\delta u_1}{\delta \sigma} + \frac{du_2}{d\sigma} \frac{\delta u_2}{\delta \sigma} + \dots + \frac{du_p}{d\sigma} \frac{\delta u_p}{\delta \sigma}.$$

If $n = 3$, this reduces to (21a), i.e., the angle θ is the same as in the corresponding three-dimensional ordinary space.

3. **Geodesics.** These curves take the place of right lines, whence their importance in non-euclidean geometry. To better understand their definition, which will be given presently, let us consider the integral

$$(25) \quad A = \int_{\alpha}^{\beta} \varphi(x, y, z, u, v) dp$$

taken over the curve C whose equations are $x = x(p)$, $y = y(p)$, $z = z(p)$. Here u, v are functions of p, x, y, z and their derivatives. Let us in this integral replace x, y, z by $\bar{x} = x + \delta x$, $\bar{y} = y + \delta y$, $\bar{z} = z + \delta z$. Geometrically speaking we replace C by an adjacent curve having however the same endpoints. At the same time u becomes $u + \delta u$, v becomes $v + \delta v$, while φ becomes $\bar{\varphi} = \varphi(x + \delta x, \dots u + \delta u, v + \delta v)$, which, developed by Taylor's theorem, gives

$$\delta \varphi = \bar{\varphi} - \varphi = \frac{\partial \varphi}{\partial x} \delta x + \dots + \frac{\partial \varphi}{\partial v} \delta v,$$

neglecting small quantities beyond the first order. Then

$$\delta A = \bar{A} - A = \int_{\alpha}^{\beta} \bar{\varphi} dp - \int_{\alpha}^{\beta} \varphi dp = \int_{\alpha}^{\beta} \delta \varphi dp.$$

When the original curve C is such that $\delta A = 0$, we say the curve renders the integral (25) *stationary*. Ordinarily it corresponds to a maximum or minimum value of A .

Let us apply these considerations to the integral

$$s = \int_{\alpha}^{\beta} \sqrt{\sum a_{ij} \frac{dx_i}{dp} \frac{dx_j}{dp}} dp = \int ds,$$

which gives the length of an arc of the curve (11). If this curve is such that

$$(26) \quad \delta \int ds = 0,$$

we say that it is a *geodesic*, ordinarily it is the shortest curve between the two fixed points $p = \alpha$, $p = \beta$. The variational equation (26) leads easily to the n equations

$$(27) \quad \sum_{i,j} \frac{dx_i}{ds} \frac{dx_j}{ds} \frac{\partial a_{ij}}{\partial x_k} - 2 \sum_i \frac{d}{ds} \left(a_{ik} \frac{dx_i}{ds} \right) = 0, \quad k = 1, 2, \dots n.$$

Example 7. In case $n = 2$, these equations become, on setting $x_1 = u$, $x_2 = v$,

$$(28) \quad \begin{aligned} 2 \frac{d}{ds} \left[a_{11} \frac{du}{ds} + a_{12} \frac{dv}{ds} \right] &= \frac{\partial a_{11}}{\partial u} \left(\frac{du}{ds} \right)^2 + 2 \frac{\partial a_{12}}{\partial u} \frac{du}{ds} \frac{dv}{ds} + \frac{\partial a_{22}}{\partial u} \left(\frac{dv}{ds} \right)^2, \\ 2 \frac{d}{ds} \left[a_{21} \frac{du}{ds} + a_{22} \frac{dv}{ds} \right] &= \frac{\partial a_{11}}{\partial v} \left(\frac{du}{ds} \right)^2 + 2 \frac{\partial a_{12}}{\partial v} \frac{du}{ds} \frac{dv}{ds} + \frac{\partial a_{22}}{\partial v} \left(\frac{dv}{ds} \right)^2 \end{aligned}$$

It should be noticed that, if one of these equations is satisfied, the other is also.

Example 8. Let us consider as a special case a surface of revolution,

$$x = v \cos u, \quad y = v \sin u, \quad z = \varphi(v),$$

using the ordinary definition of $ds^2 = dx^2 + dy^2 + dz^2$. Then for an arc on this surface $ds^2 = v^2 du^2 + (1 + \psi) dv^2$, where $\psi = (d\varphi/dv)^2$.

From (28) we can show at once that the meridians $u = \text{constant}$ are geodesics on this surface. For along a meridian $du/ds = 0$, also $a_{12} = 0$ and $\partial a_{22}/\partial u = 0$. Hence both sides of the first equation of (28) are identically zero. Thus $u = \text{constant}$ is a solution of our differential equations. The parallels, $z = \text{constant}$, are the orthogonal trajectories to these geodesics.

Reverting to the general case we notice that the equations (27) involve the second derivatives of the coördinates x_i . In order to solve the equations with respect to these quantities we introduce the *symbols of Christoffel* which pervade Einstein's theory. They are

$$(29) \quad \left[\begin{array}{cc} \alpha & \beta \\ & k \end{array} \right] = \frac{1}{2} \left(\frac{\partial a_{\alpha k}}{\partial x_\beta} + \frac{\partial a_{\beta k}}{\partial x_\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x_k} \right) \quad \alpha, \beta, k = 1, 2, \dots, n,$$

$$(30) \quad \left\{ \begin{array}{cc} \alpha & \beta \\ & \lambda \end{array} \right\} = \sum_k a^{\lambda k} \left[\begin{array}{cc} \alpha & \beta \\ & k \end{array} \right] \quad k = 1, 2, \dots, n.$$

It is important to notice that they are symmetric in α, β .

Example 9. $ds^2 = x_2^2 dx_1^2 + dx_2^2$ (element of arc in polar coördinates). Here $a_{11} = x_2^2$, $a_{12} = a_{21} = 0$, $a_{22} = 1$, $a = x_2^2$, $a^{11} = 1/x_2^2$, $a^{12} = a^{21} = 0$, $a^{22} = 1$.

$$\left[\begin{array}{cc} 1 & 1 \\ & 1 \end{array} \right] = \frac{1}{2} \frac{\partial a_{11}}{\partial x_1} = 0, \quad \left[\begin{array}{cc} 1 & 2 \\ & 1 \end{array} \right] = \frac{1}{2} \frac{\partial a_{11}}{\partial x_2} = x_2,$$

$$\left[\begin{array}{cc} 2 & 2 \\ & 1 \end{array} \right] = \frac{\partial a_{12}}{\partial x_2} - \frac{1}{2} \frac{\partial a_{22}}{\partial x_1} = 0,$$

$$\left[\begin{array}{cc} 1 & 1 \\ & 2 \end{array} \right] = -x_2, \quad \left[\begin{array}{cc} 1 & 2 \\ & 2 \end{array} \right] = 0, \quad \left[\begin{array}{cc} 2 & 2 \\ & 2 \end{array} \right] = 0,$$

$$\left\{ \begin{array}{cc} 1 & 1 \\ & 1 \end{array} \right\} = a^{11} \left[\begin{array}{cc} 1 & 1 \\ & 1 \end{array} \right] + a^{21} \left[\begin{array}{cc} 1 & 1 \\ & 2 \end{array} \right] = 0,$$

$$\left\{ \begin{array}{cc} 1 & 2 \\ & 1 \end{array} \right\} = a^{11} \left[\begin{array}{cc} 1 & 2 \\ & 1 \end{array} \right] + a^{21} \left[\begin{array}{cc} 1 & 2 \\ & 2 \end{array} \right] = \frac{1}{x_2},$$

$$\left\{ \begin{smallmatrix} 2 & 2 \\ 1 & \end{smallmatrix} \right\} = 0, \quad \left\{ \begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} \right\} = -x_2, \quad \left\{ \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} \right\} = 0, \quad \left\{ \begin{smallmatrix} 2 & 2 \\ 2 & \end{smallmatrix} \right\} = 0.$$

In terms of the symbols $\{i_{\lambda}^j\}$ we may write the n equations (27)

$$(31) \quad \frac{d^2 x_{\lambda}}{ds^2} + \sum_{ij} \left\{ \begin{smallmatrix} i & j \\ \lambda \end{smallmatrix} \right\} \frac{dx_i}{ds} \frac{dx_j}{ds} = 0, \quad \lambda = 1, 2, \dots n.$$

These are the equations of a geodesic employed by Einstein. As he supposes that a body moving freely in a gravitational field describes a geodesic, these are the equations of motion of this body as, for example, Mercury about the sun (here $n = 4$). They depend entirely upon ds , that is, the metric of the surrounding space.

Example 10. Let $ds^2 = \sum a_{ij} dx_i dx_j$, the coefficients a_{ij} being constant. From (29) we see all the symbols $[^a_k{}^b] = 0$ as the a 's are constant. Thus by (30) all the $\{^a_{\lambda}{}^b\} = 0$. Hence (31) reduces to the n equations $d^2 x_{\lambda}/ds^2 = 0$, $\lambda = 1, 2, \dots n$. Integrating we get $x_{\lambda} = A_{\lambda}s + B_{\lambda}$, A_{λ} and B_{λ} being constants. These are the equations of a right line, the parameter being s . Thus when the coefficients a_{ij} which define ds^2 are constants, the geodesics in this space are right lines. This is the case in the restricted theory of relativity (6), since there the velocity c of light in vacuo is constant.

4. Elliptic space. As we shall see, Einstein assumes that our space is not infinite in extent. It has a definite volume like a sphere, viz., $V = \pi^2 R^3$, where R has the approximate value $R = 9 \cdot 10^{11}$ orbrads, this unit being the mean distance of the earth from the sun, i.e., 1 orbrad = 150 million kilometers. All geodesics (pseudo right lines) are closed curves and have the length πR . Thus, were it not for the absorption of light in traversing such enormous distances, to the sun should correspond another sun, a sort of anti-sun, in the opposite direction. Such a space may seem preposterous to the naïve mind, but so did the existence of people living at the antipodes a few hundred years ago. The first to study an elliptic space R_n , $n > 2$, was Riemann; a 2-way space of this type has been known since the days of the Greeks, it is the surface of a sphere.

Without going into details let us show how the properties of this space may be easily deduced. To this end we take a set of rectangular axes in the euclidean plane and define the position of a point by the coördinates x, y measured in the ordinary way. The distance ds between the point x, y and the point $x + dx, y + dy$ we define by

$$(32) \quad ds^2 = \frac{dx^2 + dy^2}{\left[1 + \frac{x^2 + y^2}{4R^2}\right]^2} = \frac{16R^4}{\lambda^2} (dx^2 + dy^2), \quad \lambda = x^2 + y^2 + 4R^2.$$

The metric of this R_2 is not euclidean; but we may refer it to a euclidean R_3 as follows. Set

$$(33) \quad u = \frac{4R^2x}{\lambda}, \quad v = \frac{4R^2y}{\lambda}, \quad w = \frac{8R^5}{\lambda} - R.$$

Then

$$du = \frac{4R^2}{\lambda^2} (\lambda dx - x d\lambda), \quad dv = \frac{4R^2}{\lambda^2} (\lambda dy - y d\lambda), \quad dw = -\frac{8R^3 d\lambda}{\lambda^2},$$

from which follows

$$(34) \quad ds^2 = du^2 + dv^2 + dw^2.$$

Thus to each point x, y in the elliptic plane corresponds a point u, v, w in our ordinary three-dimensional space. This illustrates the important theorem: *If the metric of an R_n is defined by*

$$ds^2 = \sum a_{ij} dx_i dx_j, \quad i, j = 1, 2, \dots, n,$$

we may choose $m + n$ new variables $u_1, \dots, u_{m+n}$ such that

$$ds^2 = du_1^2 + du_2^2 + \dots + du_{m+n}^2;$$

moreover $m \leq n(n-1)/2$.

Thus we may regard the n -way space R_n as embedded in an $(m+n)$ -way euclidean space. From (33) we find that

$$(35) \quad u^2 + v^2 + w^2 = R^2.$$

Thus when x, y ranges over the elliptic plane, the point u, v, w ranges over a sphere.

Let us now see what conclusions we can draw relative to the geometry of this plane R_2 . In the first place we find

$$x = \frac{2Ru}{R+w}, \quad y = \frac{2Rv}{R+w}.$$

To each point u, v, w corresponds a single point x, y with one exception, viz., when $R+w=0$. But then $u=v=0$, as (35) shows. The correspondence between R_2 and R_3 is thus 1 to 1 with this one exception.

The geodesics or, as we shall call them, the *pseudo right lines*, are determined by

$$\delta \int ds = 0,$$

where ds is defined by (32). If we change to the u, v, w variables, ds is defined by (34) subject, however, to the relation (35). Thus, to pseudo right lines correspond geodesics on the sphere (35), i.e., to great circles on this sphere. From this we have:

- (i) All pseudo right lines in this R_2 are closed curves.
- (ii) Their length is $2\pi R$.

- (iii) Two pseudo right lines meet in two points. Hence
- (iv) There are no parallels.

Let C_1, C_2 be two curves making an angle θ with each other as defined by (21). On changing to the u, v, w variables these curves go over into two curves Γ_1, Γ_2 on the sphere, and $ds^2 = du^2 + dv^2 + dw^2$. But in this case we saw that θ is the angle between Γ_1 and Γ_2 in the ordinary way. Hence we have

- (v) The trigonometry of our R_2 is the trigonometry on a sphere of radius R . The sum of the angles of a triangle formed by three pseudo right lines is always greater than 180° .

Since all great circles on a sphere perpendicular to a given great circle meet at a point, viz., the pole of this circle, and hence have the length $\pi R/2$, we have

- (vi) All pseudo right lines in the elliptic plane perpendicular to a given pseudo right line meet at a point and have a common length $\pi R/2$.

We have so far made no attempt to visualize the pseudo right lines in the elliptic plane. It is easy to do this; for on the sphere they correspond to great circles. Let one of these great circles lie in the plane $Au + Bv + Cw = 0$. Replacing u, v, w by their values in (33) we get

$$ARx + BRy - (x^2 + y^2 + 4R^2) + 8R^2C = 0,$$

the equation of a circle in the (euclidean) x, y plane. In particular, the pseudo right line corresponding to the equation $w = 0$ is the circle

$$(36) \quad x^2 + y^2 = 4R^2,$$

which we call the *fundamental circle*. Since all great circles cut the equator in diametrically opposite points, we see that all pseudo right lines cut the fundamental circle in such points. Conversely, such circles are pseudo right lines in our elliptic geometry.

The geometry so far developed differs from plane euclidean geometry therein that its pseudo right lines cut a given pseudo right line twice. We may, if we like, agree to regard all points of the x, y plane outside the fundamental circle as non-existent as far as our elliptic geometry is concerned. Also we shall assume that diametrically opposite points of this circle are identical. In this case two pseudo right lines cut once only and they all have the common length πR .

Let us turn now to elliptic space; as the work is entirely analogous, we may be more brief. We start with a set of rectangular coördinates and define a point by the coördinates x, y, z measured in the ordinary way. We define the metric by

$$(37) \quad ds^2 = \frac{dx^2 + dy^2 + dz^2}{\left[1 + \frac{1}{4R^2}(x^2 + y^2 + z^2)\right]^2} = \frac{16R^2(dx^2 + dy^2 + dz^2)}{[x^2 + y^2 + z^2 + 4R^2]^2}.$$

As before we set $\lambda = x^2 + y^2 + z^2 + 4R^2$ and

$$(38) \quad u_1 = \frac{4R^2x}{\lambda}, \quad u_2 = \frac{4R^2y}{\lambda}, \quad u_3 = \frac{4R^2z}{\lambda}, \quad u_4 = \frac{8R^3}{\lambda} - R,$$

and find again that

$$ds^2 = du_1^2 + du_2^2 + du_3^2 + du_4^2.$$

In an entirely analogous manner we find that geodesics in this space or, as we prefer to call them, pseudo right lines cut the fundamental sphere

$$(39) \quad x^2 + y^2 + z^2 = 4R^2$$

in diametrically opposite points.

The analogue of the euclidean plane is a sphere cutting the fundamental sphere along a great circle. We call it a pseudo plane. As before we have two geometries according as we regard opposite ends of a diameter of the fundamental sphere (39) as identical or not. In the former case points outside of (39) are non-existent. Einstein in his cosmological considerations prefers this type of geometry. In this sphere we have:

- (i) All pseudo right lines have the length πR and are closed curves.
- (ii) These lines cut once only.
- (iii) There are no parallels.
- (iv) Two points determine a pseudo right line.
- (v) Three points determine a pseudo plane.

According to this geometry the whole physical universe lies within the fundamental sphere. Let us find the volume. By (23)

$$(40) \quad V = \int \sqrt{a} \, dx dy dz.$$

From (37) we have, setting $\alpha^{-1} = 1 + \rho^2(x^2 + y^2 + z^2)$, $\rho = 1/2R$,

$$a = \begin{vmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{vmatrix} = \alpha^6.$$

Let us change the variables in the integral (40), setting

$$x = r \cos \theta \cos \varphi, \quad y = r \cos \theta \sin \varphi, \quad z = r \sin \theta.$$

Then

$$V = \int \left| J \right| \frac{dr d\theta d\varphi}{R},$$

where

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix}.$$

Thus

$$(41) \quad V = \int_0^{2R} \frac{r^2 dr}{(1 + \rho^2 r^2)^3} \int_{-(\pi/2)}^{\pi/2} \cos \theta d\theta \int_0^{2\pi} d\varphi = 4\pi \int_0^{2R} \frac{r^2 dr}{(1 + \rho^2 r^2)^3} = \pi^2 R^3.$$

As we have remarked and as we shall see later, an approximate value of R is $9 \cdot 10^{11}$ times the mean distance of the earth from the sun.

5. **Curvature.** The metric properties of a given R_n depend, as we have seen, on the definition of distance between two nearby points, i.e., on the quadratic form

$$(42) \quad ds^2 = \sum a_{ij} dx_i dx_j.$$

Another space R_n' , whose metric is defined by quite a different expression

$$(43) \quad d\sigma^2 = \sum b_{ij} du_i du_j,$$

may have essentially the same geometry. For example, in R_2 let $ds^2 = dx_1^2 + dx_2^2$, and in R_2' let $d\sigma^2 = u_2^2 du_1^2 + du_2^2$. If we set

$$(44) \quad x_1 = u_2 \cos u_1, \quad x_2 = u_2 \sin u_1,$$

we find $ds^2 = d\sigma^2$. The relations (44) enable us to establish a 1 to 1 correspondence between the points of R_2 and R_2' . Since $ds = d\sigma$, corresponding arcs have the same length and corresponding angles are equal. Hence their metrical properties are the same.

We are therefore led to ask when is it possible, by a suitable change of variables, to transform (42) into (43), and conversely. Without answering this with entire generality we may give a partial answer sufficient for our purpose. To this end we introduce the *symbols of Riemann*.

$$(45) \quad \{\alpha\beta, \lambda\mu\} = \frac{\partial \left\{ \begin{smallmatrix} \alpha & \lambda \\ & \beta \end{smallmatrix} \right\}}{\partial x_\mu} - \frac{\partial \left\{ \begin{smallmatrix} \alpha & \mu \\ & \beta \end{smallmatrix} \right\}}{\partial x_\lambda} \\ + \sum \left[\left\{ \begin{smallmatrix} \alpha & \lambda \\ & k \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} k & \mu \\ & \beta \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} \alpha & \mu \\ & k \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} k & \lambda \\ & \beta \end{smallmatrix} \right\} \right], \quad k=1, 2, \dots, n,$$

and

$$(46) \quad (\alpha \gamma, \lambda \mu) = \sum_\beta a_{\beta\gamma} \{\alpha \beta, \lambda \mu\}.$$

As in the case of the Christoffel symbols we have

$$(47) \quad \{\alpha, \beta, \lambda, \mu\} = \sum_\gamma a^{\beta\gamma} (\alpha \gamma, \lambda \mu).$$

By means of (46) we may separate out an important class of n -way spaces called *spaces of constant curvature*.^{*} We say R_n has constant curvature k when for all $\alpha, \beta, \lambda, \mu = 1, 2, \dots, n$

$$(48) \quad (\alpha \beta, \lambda \mu) = k \begin{vmatrix} a_{\alpha\lambda} & a_{\alpha\mu} \\ a_{\beta\lambda} & a_{\beta\mu} \end{vmatrix}.$$

We may now state the important theorem: *If R_n, R_n' are two spaces of the same constant curvature k , we may transform (42) into (43) by a suitable change of variable, and conversely; that is, $ds = d\sigma$. The metric properties of the two spaces are the same, at least for sufficiently restricted regions.*

Riemann showed that for spaces of constant curvature k the element of arc may be defined by

^{*} As the term *curvature* figures so largely in Einstein's theory and quite wrong ideas are current in some quarters, a few additional words of explanation may be acceptable. In ordinary space the curvature of a surface S at a point x is defined by

$$(a) \quad k = \frac{1}{R_1 R_2},$$

where R_1, R_2 are the greatest and least radii of curvature of the normal sections of S at x . Gauss made the extraordinary discovery that k remains invariant under all transformations of the variables. We find in fact that, if the metric of S is given by

$$(b) \quad ds^2 = a_{11}du_1^2 + 2a_{12}du_1du_2 + a_{22}du_2^2,$$

then

$$(c) \quad k = \frac{(12, 12)}{a}.$$

Suppose now the surface S , lying in an n -way space R_n , is defined by $x_1 = x_1(u_1, u_2), \dots, x_n = x_n(u_1, u_2)$. If the metric of R_n is defined by (13), the element of arc $d\sigma$ on S is given by

$$d\sigma^2 = \sum_{i,j} a_{ij} \sum_k \frac{\partial x_i}{\partial u_k} du_k \sum_l \frac{\partial x_j}{\partial u_l} du_l, \quad i, j = 1, 2, \dots, n; \quad k, l = 1, 2,$$

or

$$(d) \quad d\sigma^2 = g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2.$$

The curvature of S at a point x is now defined by

$$(e) \quad k = \frac{(12, 12)_u}{g},$$

where g is the determinant of the quadratic form (d) and $(12, 12)_u$ is the Riemannian symbol (46) relative to this form. We see this definition is merely an extension of (c) from 3 to n -way space. But, whereas (c) has the geometric interpretation (a), the definition (e) has not, it is merely an analytic generalization. The reader should not undervalue it on that score; its importance is fundamental.

Let us now consider a curve C . The n quantities $\eta_1 = dx_1/ds, \dots, \eta_n = dx_n/ds$ are called the *directional parameters* of C at a given point x . Through any point x of our R_n there passes a geodesic having a given $\eta = (\eta_1, \eta_2, \dots, \eta_n)$. If $\eta' = (\eta'_1, \dots, \eta'_n)$ is another set of parameters at x , $g\eta + g'\eta'$ will denote a pencil of these parameters, g, g' being variables. To this pencil corresponds a pencil of geodesics through x having $g\eta + g'\eta'$ as directional parameters. These geodesics constitute a surface G in R_n on which an element of arc $d\sigma^2$ has the form (d). The curvature k of G at the point x is given by (e). Suppose now that k has the same constant value however the pencil (η, η') is oriented about x , we say R_n is a *space of constant curvature* k .

$$(49) \quad ds^2 = \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2}{[1 + \frac{1}{4}k^2(x_1^2 + x_2^2 + \cdots + x_n^2)]^2}.$$

When $k = 0$, this reduces to $ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2$. In this case we have seen that the geodesics are right lines, the $x_1, \cdots, x_n$ being referred to a rectangular coördinate system (for clearness the reader may suppose $n = 3$). We therefore regard k as a measure of the departure of the space R_n defined by (42) from euclidean space. Thus we saw in the elliptic space R_3 that the geodesics, instead of being straight lines, are arcs of circles. Here $k = 1/R^2$, as is seen by comparing (37) and (49). The smaller k is, the more nearly these geodesics or pseudo right lines approach straight lines in euclidean space.

Example 11. Let us see if the R_2 whose metric is defined by

$$(50) \quad ds^2 = c^2 \cos^2 x_2 dx_1^2 + c^2 dx_2^2$$

has constant curvature. Here

$$a_{11} = c^2 \cos^2 x_2, \quad a_{12} = a_{21} = 0, \quad a_{22} = c^2, \quad a = c^4 \cos^2 x_2,$$

$$a^{11} = \frac{1}{a_{11}}, \quad a^{12} = a^{21} = 0, \quad a^{22} = \frac{1}{a_{22}},$$

$$\begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} = 0, \quad \begin{bmatrix} 1 & 2 \\ 1 & \end{bmatrix} = -c^2 \sin x_2 \cos x_2, \quad \begin{bmatrix} 2 & 2 \\ 1 & \end{bmatrix} = 0,$$

$$\begin{bmatrix} 1 & 1 \\ 2 & \end{bmatrix} = c^2 \sin x_2 \cos x_2, \quad \begin{bmatrix} 1 & 2 \\ 2 & \end{bmatrix} = 0, \quad \begin{bmatrix} 2 & 2 \\ 2 & \end{bmatrix} = 0,$$

$$\left\{ \begin{array}{l} 1 & 1 \\ 1 & \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} 1 & 2 \\ 1 & \end{array} \right\} = -\tan x_2, \quad \left\{ \begin{array}{l} 2 & 2 \\ 1 & \end{array} \right\} = 0,$$

$$\left\{ \begin{array}{l} 1 & 1 \\ 2 & \end{array} \right\} = \sin x_2 \cos x_2, \quad \left\{ \begin{array}{l} 1 & 2 \\ 2 & \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} 2 & 2 \\ 2 & \end{array} \right\} = 0,$$

$$(1 \ 2, 1 \ 2) = a_{12}\{1 \ 1, 1 \ 2\} + a_{22}\{1 \ 2, 1 \ 2\} = c^2\{1 \ 2, 1 \ 2\},$$

$$\begin{aligned} \{1 \ 2, 1 \ 2\} &= \frac{\partial}{\partial x_2} \left\{ \begin{array}{l} 1 & 1 \\ 2 & \end{array} \right\} - \left\{ \begin{array}{l} 1 & 2 \\ 1 & \end{array} \right\} \left\{ \begin{array}{l} 1 & 1 \\ 2 & \end{array} \right\} \\ &= -\sin^2 x_2 + \cos^2 x_2 + \tan x_2 \sin x_2 \cos x_2 = \cos^2 x_2, \\ \therefore (1 \ 2, 1 \ 2) &= c^2 \cos^2 x_2. \end{aligned}$$

From (48)

$$(1 \ 2, 1 \ 2) = k \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = ak, \quad \therefore k = \frac{1}{c^2}.$$

By using the fact that $(\alpha \beta, \lambda \mu)$ changes its sign when we interchange α, β or λ, μ , and hence is zero when $\alpha = \beta$ or $\lambda = \mu$, we find that all the $2^4 = 16$ symbols $(\alpha \beta, \lambda \mu)$ placed in (48) either give $k = 1/c^2$ or $0 = k \cdot 0$. Thus the 16 relations (48) are satisfied by this value of k . Hence R_2 as defined by (50) is a 2-way space of constant curvature $k = 1/c^2$. In

fact if we regard x_1, x_2 as longitude and latitude, (50) defines ds on a sphere of radius c .

Example 12.

$$(51) \quad ds^2 = c_1 dx_1^2 + c_2 dx_2^2 + \cdots + c_n dx_n^2,$$

where the coefficients are constants. Here all the $[\alpha_k^\beta] = 0$ since the a_{ij} are constants, a_{ij} being zero if $i \neq j$. Hence all the $\{\alpha_k^\beta\} = 0$. Thus all the $\{\alpha\beta, \lambda\mu\} = 0$, and hence finally all the $(\alpha\beta, \lambda\mu) = 0$. Thus the n^4 equations (48) are satisfied by $k = 0$. The curvature of the space defined by (50) is therefore 0. A special case of (51) is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2,$$

which defines the metric of the 4-way space of the restricted theory of relativity. Although we call a space for which $k = 0$ euclidean, the reader should note that such a space may possess pseudo lines of null length, i.e., lines for which $ds = 0$. If we set $ds = 0$ in the last equation, we get

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0,$$

which is (4). Thus the path of a ray of light is a null line in the restricted theory of relativity.

6. Tensors. To form invariant differential equations expressing the laws of physics, Einstein found ready at hand a calculus which seems almost created for his needs. This is the *calcul différentiel absolu* of Ricci and Levi-Civita already referred to. We think a better name is *tensor analysis*. To give the reader a concrete example of a tensor, in fact one of the most important tensors, let us see how the n^2 coefficients a_{ij} in

$$(52) \quad ds^2 = \sum_{i,j} a_{ij} dx_i dx_j, \quad i, j = 1, 2, \dots, n,$$

behave when we replace the variables $x_1, \dots, x_n$ by n new variables $u_1, \dots, u_n$. Since

$$dx_i = \sum_{\lambda} \frac{\partial x_i}{\partial u_{\lambda}} du_{\lambda}, \quad \lambda = 1, 2, \dots, n,$$

(52) becomes

$$ds^2 = \sum_{i,j} a_{ij} \sum_{\lambda} \frac{\partial x_i}{\partial u_{\lambda}} du_{\lambda} \sum_{\mu} \frac{\partial x_j}{\partial u_{\mu}} du_{\mu} = \sum_{\lambda, \mu} du_{\lambda} du_{\mu} \sum_{i,j} a_{ij} \frac{\partial x_i}{\partial u_{\lambda}} \frac{\partial x_j}{\partial u_{\mu}}.$$

Hence

$$ds^2 = \sum \bar{a}_{\lambda\mu} du_{\lambda} du_{\mu}, \quad \lambda, \mu = 1, 2, \dots, n,$$

where

$$(53) \quad \bar{a}_{\lambda\mu} = \sum_{i,j} \frac{\partial x_i}{\partial u_{\lambda}} \frac{\partial x_j}{\partial u_{\mu}} a_{ij}.$$

Let us generalize and say that the n^2 functions A_{ij} of $x_1, \dots, x_n$ form

a covariant tensor of order 2 if, on changing the variables to $u_1, \dots, u_n$, the transformed $\bar{A}_{\lambda\mu}$ are related to the old A_{ij} by

$$(54) \quad \bar{A}_{\lambda\mu} = \sum_{i,j} \frac{\partial x_i}{\partial u_\lambda} \frac{\partial x_j}{\partial u_\mu} A_{ij}.$$

The individual A_{ij} are called the components of this tensor. From this we see the n^2 coefficients a_{ij} in (52) form a covariant tensor of order 2.

We may generalize (54) as follows. Suppose we have n^k functions $A_{\alpha\beta\dots\kappa}$ of $x_1, \dots, x_n$ which are transformed according to

$$(55) \quad \bar{A}_{\lambda\mu\dots\omega} = \sum \frac{\partial x_\alpha}{\partial u_\lambda} \frac{\partial x_\beta}{\partial u_\mu} \dots \frac{\partial x_\kappa}{\partial u_\omega} A_{\alpha\beta\dots\kappa},$$

the summation extending over the k indices $\alpha, \beta, \dots, \kappa$ from 1 to n . We say these n^k functions form a *covariant tensor of order k* .

Example 13. The n^4 symbols of Riemann ($\alpha\beta, \gamma\delta$) are transformed in this way. In fact, if we set

$$(56) \quad G_{\alpha\beta\gamma\delta} = (\alpha\beta, \gamma\delta),$$

we find by a reasoning too long to give here that, on changing variables,

$$(57) \quad \bar{G}_{\lambda\mu\nu\omega} = \sum_{\alpha,\beta,\gamma,\delta} \frac{\partial x_\alpha}{\partial u_\lambda} \frac{\partial x_\beta}{\partial u_\mu} \frac{\partial x_\gamma}{\partial u_\nu} \frac{\partial x_\delta}{\partial u_\omega} G_{\alpha\beta\gamma\delta}, \quad \alpha, \beta, \gamma, \delta = 1, 2, \dots, n.$$

The reader should note that the new variables in (55) are in the denominator and that their k indices are those of the transformed component $\bar{A}_{\lambda\mu\dots\omega}$.

7. Contravariant tensors. If the n^2 functions A^{ij} of $x_1, \dots, x_n$ on changing the variables to $u_1, \dots, u_n$ go over into

$$(58) \quad \bar{A}^{\lambda\mu} = \sum_{i,j} \frac{\partial u_\lambda}{\partial x_i} \frac{\partial u_\mu}{\partial x_j} A^{ij},$$

we say they form a *contravariant tensor of order 2*. If we compare (54) with (58), we see the relations between the old and the transformed components differ by having the new variables u in the denominator when covariant and in the numerator when contravariant. The extension of (58) to define contravariant tensors of order k is obvious; instead of 2 partial derivatives we have k .

Example 14. Set $A^1 = dx_1, A^2 = dx_2, \dots, A^n = dx_n$. The reader should note that 2 in A^2 is an upper index and not an exponent, and so on. On changing the variables these become

$$\bar{A}^1 = du_1, \quad \bar{A}^2 = du_2, \quad \dots \quad \bar{A}^n = du_n.$$

But

$$(59) \quad \bar{A}^\lambda = du_\lambda = \sum_i \frac{\partial u_\lambda}{\partial x_i} dx_i = \sum_i \frac{\partial u_\lambda}{\partial x_i} A^i.$$

Hence the n quantities form a contravariant tensor of order 1.

Example 15. By a reasoning unfortunately too long to give here it can be shown that the n^2 quantities a^{ij} defined in (19) form a contravariant tensor of order 2.

8. Mixed tensors. Suppose the law of transformation of the n^2 functions of $x_1, \dots, x_n$, call them A_β^α , is defined by

$$(60) \quad \bar{A}_\beta^\alpha = \sum_{i,j} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial x_j}{\partial u_\beta} A_j^i.$$

If there were l factors with u in the numerator and m factors with u in the denominator, the tensor would be a mixed tensor of order $l + m$, covariant of order m , contravariant of order l .

Example 16. Let us show that the n^4 functions of $x_1, \dots, x_n$

$$(61) \quad G_{\alpha\lambda\mu}^\beta = \{\alpha\beta, \lambda\mu\} = \sum_\gamma a^{\beta\gamma} G_{\alpha\gamma\lambda\mu}$$

are the components of a mixed tensor, covariant of order 3 and contravariant of order 1. For, by (57) and (58), we have

$$\begin{aligned} \bar{G}_{\alpha\lambda\mu}^\beta &= \sum_\gamma \sum_{i,j} \frac{\partial u_\beta}{\partial x_i} \frac{\partial u_\gamma}{\partial x_j} a^{ij} \sum_{h,k,r,s} \frac{\partial x_h}{\partial u_\alpha} \frac{\partial x_k}{\partial u_\lambda} \frac{\partial x_r}{\partial u_\mu} \frac{\partial x_s}{\partial u_\gamma} G_{hkr s} \\ &= \sum_{i,j,h,k,r,s} (\dots) \sum_\gamma \frac{\partial u_\gamma}{\partial x_j} \frac{\partial x_k}{\partial u_\gamma}. \end{aligned}$$

From the calculus we know that

$$\sum_\gamma \frac{\partial u_\gamma}{\partial x_j} \frac{\partial x_k}{\partial u_\gamma} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Thus the terms in the sum over i, j, k, r, s drop out for which $j \neq k$; therefore

$$\begin{aligned} \bar{G}_{\alpha\lambda\mu}^\beta &= \sum_{i,h,r,s} \frac{\partial u_\beta}{\partial x_i} \frac{\partial x_h}{\partial u_\alpha} \frac{\partial x_r}{\partial u_\lambda} \frac{\partial x_s}{\partial u_\mu} \sum_j a^{ij} G_{hjr s} \\ (62) \quad &= \sum_{i,h,r,s} \frac{\partial u_\beta}{\partial x_i} \frac{\partial x_h}{\partial u_\alpha} \frac{\partial x_r}{\partial u_\lambda} \frac{\partial x_s}{\partial u_\mu} G_{hrs}^i, \quad \text{q.e.d.} \end{aligned}$$

We must note one highly important feature which all tensors have in common: The components of the transformed tensor are always *linear* in the components of the original tensor (cf. (53), (55), (57), (58), (59), (60), (61)).

Suppose now a certain law in physics is expressed by the vanishing of the components of a certain tensor, say, for example, by

$$(63) \quad A_{\alpha\beta} = 0. \quad \alpha, \beta = 1, 2, \dots, n.$$

If we introduce the new variables $u_1, \dots, u_n$, these n^2 equations go over into

$$\bar{A}_{\alpha\beta} = \sum \frac{\partial x_i}{\partial u_\alpha} \frac{\partial x_j}{\partial u_\beta} A_{ij}.$$

But, as each $A_{ij} = 0$ by hypothesis, we have

$$(64) \quad \bar{A}_{\alpha\beta} = 0.$$

Thus the equations (63) hold for any set of coördinates, that is, they are invariant.

9. Operations on tensors. The *sum* of two tensors of like character A, B is a tensor whose components are the sum of the components of A and B . Thus the sum of $A = \{A_{\alpha\beta}\}$ and $B = \{B_{\alpha\beta}\}$ has the components $A_{\alpha\beta} + B_{\alpha\beta}$.

The *product* of two tensors, as $A = \{A_{\alpha\beta}\}$ and $B = \{B^i\}$, is the tensor whose components are $C_{\alpha\beta}{}^\gamma = A_{\alpha\beta} B^\gamma$.

The *composition* of two tensors is best illustrated by an example. Suppose $A = \{A_\alpha{}^{\lambda\mu}\}$, $B = \{B_{\lambda\mu\nu}^{\alpha\delta\epsilon\eta}\}$; we set

$$(65) \quad C_\nu{}^{\delta\epsilon\eta} = \sum_{\alpha, \lambda, \mu} A_\alpha{}^{\lambda\mu} B_{\lambda\mu\nu}^{\alpha\delta\epsilon\eta},$$

the sum extending over the common indices, which must be upper indices in the one tensor and lower in the other. It can be shown easily that the result is a tensor whose character is obtained by cancelling these common indices as indicated in (65).

Example 17. Let $A = \{a_{ij}\}$, $B = \{a^{ir}\}$. Their composition gives

$$(66) \quad \sum_i a_{ij} a^{ir} = a_j{}^r = \begin{cases} 1 & \text{if } r = j, \\ 0 & \text{if } r \neq j, \end{cases}$$

as we saw in (20).

Example 18. The composition of this mixed tensor $\{a_j{}^r\}$ with the mixed tensor $\{G_{\alpha r}{}^\mu{}^j\}$ defined in (61) gives the tensor whose components are

$$(67) \quad \sum_{r,j} a_j{}^r G_{\alpha r}{}^\mu{}^j = \sum_r G_{\alpha r}{}^\mu{}^r = \sum_r \{\alpha r, r \mu\} = G_{\alpha\mu}.$$

This tensor is historic. In fact the equations

$$(68) \quad G_{\alpha\mu} = 0, \quad \alpha, \mu = 1, 2, 3, 4,$$

determine the metric (9) of the space about the sun.

Contraction. This is another operation which leads to a tensor. Suppose, for example, in a mixed tensor whose components are $A_{\lambda\mu\nu}{}^{\alpha\beta}$; we set $\alpha = \lambda$, $\beta = \nu$ and sum over α, β , thus

$$(69) \quad \sum_{\alpha, \beta} A_{\alpha\mu\beta}{}^{\alpha\beta} = A_\mu.$$

This is found to be a tensor whose character is obtained by dropping the common upper and lower indices. Thus (69) are the components of a covariant tensor of order 1 as indicated.

Example 19. Contracting $\{G_{\alpha r \mu}^j\}$ by setting $j = r$ we get a tensor whose components are

$$\sum_r G_{\alpha r \mu}^r = \sum_r \{\alpha r, r \mu\} = G_{\alpha \mu},$$

the same tensor obtained by composition in example 18.

10. **Tensors of order 0.** If we compound $\{A_{ij}\}$ with $\{B^{ij}\}$, we get a tensor whose sole component is

$$(70) \quad \sum_{i,j} A_{ij} B^{ij}.$$

On transforming this becomes

$$(71) \quad \begin{aligned} \sum_{i,j} \bar{A}_{ij} \bar{B}^{ij} &= \sum_{i,j} \sum_{\alpha,\beta} \frac{\partial x_\alpha}{\partial u_i} \frac{\partial u_\beta}{\partial x_j} A_{\alpha\beta} \sum_{\lambda,\mu} \frac{\partial u_i}{\partial x_\lambda} \frac{\partial u_j}{\partial x_\mu} A^{\lambda\mu} \\ &= \sum_{\alpha,\beta,\lambda,\mu} A_{\alpha\beta} A^{\lambda\mu} \sum_i \frac{\partial x_\alpha}{\partial u_i} \frac{\partial u_i}{\partial x_\lambda} \sum_j \frac{\partial x_\beta}{\partial u_j} \frac{\partial u_j}{\partial x_\mu}. \end{aligned}$$

Now $\sum_j = 0$ or 1 according as $\mu = \beta$ or does not, and a similar remark holds for the sum $\sum_i$. Thus (71) reduces to

$$\sum \bar{A}_{ij} \bar{B}^{ij} = \sum A_{ij} A^{ij},$$

i.e., the expression is an invariant. On the other hand (70) is a tensor whose character is obtained by omitting common upper and lower indices; as no index is left, we may regard it as a tensor of order 0. The foregoing may be extended obviously to tensors of any order.

Example 20.

$$ds^2 = \sum a_{ij} dx_i dx_j.$$

Here a_{ij} is covariant of order 2, $\{dx_i dx_j\} = \{B^{ij}\}$ is a contravariant tensor of order 2 also. As ds^2 is obtained by compounding these two tensors, it is an invariant.

Example 21.

$$\cos \theta = \sum_{i,j} a_{ij} \frac{dx_i}{ds} \frac{\delta x_j}{\delta s}.$$

This is the composition of $\{a_{ij}\}$ with $\left\{ \frac{dx_i}{ds} \frac{\delta x_j}{\delta s} \right\} = \{B^{ij}\}$. Hence $\cos \theta$ is an invariant, as already observed.

Example 22. Beltrami's parameter.

$$(72) \quad \Delta_1(\varphi) = \sum_{i,j} a^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j}.$$

Let us first show that $A_i = \partial \varphi / \partial x_i$ are the components of a covariant tensor of order 1. In fact

$$\bar{A}_i = \frac{\partial \varphi}{\partial u_i} = \frac{\partial \varphi}{\partial x_1} \frac{\partial x_1}{\partial u_i} + \dots + \frac{\partial \varphi}{\partial x_n} \frac{\partial x_n}{\partial u_i}; \quad \therefore \bar{A}_i = \sum_\alpha \frac{\partial x_\alpha}{\partial u_i} A_\alpha.$$

Thus $B_{ij} = \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j}$ are the components of a covariant tensor of order 2.

Hence (72) is an invariant.

Example 23. Beltrami's mixed parameter.

$$(73) \quad \nabla(\varphi, \psi) = \sum_{ij} a^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j}$$

is obviously an invariant.

Example 24.

$$\sum_{r,s} a_{rs} a^{rs}.$$

This is also an invariant. In fact by (66)

$$(74) \quad \sum_{r,s} a_{rs} a^{rs} = \sum_r \sum_s a_{rs} a^{rs} = \sum_r a_r^r = n.$$

Example 25.

$$(75) \quad G = \sum_{\lambda, \mu} a^{\lambda\mu} G_{\lambda\mu},$$

where by (67) $G_{\lambda, \mu} = \sum_h \{\lambda h, h \mu\}$ is an invariant. This invariant is fundamental in Einstein's theory, as we shall see. It is called the *curvature invariant*. For a euclidean space G is zero.

11. Covariant differentiation. Let $\{A_i\}$ be a tensor of order 1. We find that

$$(76) \quad A_{\alpha/\lambda} = \frac{\partial A_\alpha}{\partial x_\lambda} - \sum_h \left\{ \begin{matrix} \alpha & \lambda \\ & h \end{matrix} \right\} A_h$$

are the components of a covariant tensor of order 2. Similarly

$$(77) \quad A_{/\lambda}^\alpha = \frac{\partial A^\alpha}{\partial x_\lambda} + \sum_h \left\{ \begin{matrix} h & \lambda \\ & \alpha \end{matrix} \right\} A^h$$

are the components of a contravariant tensor of order 2. These tensors we say are obtained from $\{A_\alpha\}$ and $\{A^\alpha\}$ by *covariant differentiation*. It is easy to extend this process to tensors of any order. Thus the covariant derivative of the three types of tensors of order 2 relative to x_λ are the tensors of order 3 whose components are

$$(78) \quad A_{\alpha\beta/\lambda} = \frac{\partial A_{\alpha\beta}}{\partial x_\lambda} - \sum_h \left\{ \begin{matrix} \alpha & \lambda \\ & h \end{matrix} \right\} A_{h\beta} - \sum_h \left\{ \begin{matrix} \beta & \lambda \\ & h \end{matrix} \right\} A_{\alpha h},$$

$$(79) \quad A_{/\lambda}^{\alpha\beta} = \frac{\partial A^{\alpha\beta}}{\partial x_\lambda} + \sum_h \left\{ \begin{matrix} h & \lambda \\ & \alpha \end{matrix} \right\} A^{h\beta} + \sum_h \left\{ \begin{matrix} h & \lambda \\ & \beta \end{matrix} \right\} A^{\alpha h},$$

$$(80) \quad A_{\beta/\lambda}^\alpha = \frac{\partial A_\beta^\alpha}{\partial x_\lambda} - \sum_h \left\{ \begin{matrix} \beta & \lambda \\ & h \end{matrix} \right\} A_h^\alpha + \sum_h \left\{ \begin{matrix} h & \lambda \\ & \alpha \end{matrix} \right\} A_\beta^h.$$

It is important to note that the covariant derivatives of the fundamental tensor $\{a_{ij}\}$ are all zero. Let us note also that when these com-

ponents a_{ij} are constants, covariant differentiation is identical with ordinary differentiation, since the Christoffel symbols $\{\alpha_{\gamma}^{\beta}\}$ are all zero.

Example 26. Let F be a function of $x_1, \dots, x_n$ and set $F_i = \partial F / \partial x_i$. Then by (76)

$$F_{i/k} = \frac{\partial^2 F}{\partial x_i \partial x_k} - \sum_h \left\{ \begin{matrix} i & k \\ h \end{matrix} \right\} \frac{\partial F}{\partial x_h}.$$

If we compound this tensor with the tensor whose components are a^{ik} , we get *Beltrami's second differential parameter*, which is therefore an invariant; viz.,

$$(81) \quad \Delta_2 F = \sum_{i,k} a^{ik} F_{i/k} = \sum_{i,k} a^{ik} \left[\frac{\partial^2 F}{\partial x_i \partial x_k} - \sum_h \left\{ \begin{matrix} i & k \\ h \end{matrix} \right\} \frac{\partial F}{\partial x_h} \right].$$

When $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$, this reduces to

$$\Delta_2 F = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} + \frac{\partial^2 F}{\partial x_3^2}.$$

12. Divergence. In the restricted theory of relativity the divergence of certain tensors is of fundamental importance. They are equally important in Einstein's theory. Consider, for example, the covariant tensor whose components are $A_{\lambda\mu}$. Its covariant derivative relative to x_k has the components $A_{\lambda\mu/k}$. It is of order 3. To get a tensor of order 1 we compound it with the fundamental tensor $\{a^{\mu k}\}$ getting a covariant tensor of order 1 whose components are

$$(82) \quad \sum_{\mu} a^{\mu k} A_{\lambda\mu/k} = A_{\lambda};$$

we call this tensor the *divergence* of $\{A_{\lambda\mu}\}$ and write it $\text{div } \{A_{\lambda\mu}\}$. Similarly the divergence of the contravariant tensor $\{A^{\lambda\mu}\}$ relative to x_k has the components

$$(83) \quad \sum_{\mu} a_{\mu k} A^{\lambda\mu/k} = \sum_k A^{\lambda k} = A^{\lambda},$$

obviously a contravariant tensor of order 1. In a similar manner we may define the divergence of any tensor, but, as we shall not need them, we will not take space to write them down.

13. Einstein's metric. We have seen that the metric of Einstein's 4-way space is determined by a quadratic differential form

$$(84) \quad ds^2 = \sum a_{ij} dx_i dx_j, \quad i, j = 1, 2, 3, 4, \quad a_{ij} = a_{ji}.$$

As yet, however, the 10 coefficients a_{ij} are undetermined functions of the 3 space coördinates x_1, x_2, x_3 and the time coördinate x_4 . To determine these a 's Einstein makes use of the fact that the restricted relativity theory gives a very satisfactory account of a wide class of phenomena. In this theory the metric is given by

$$(85) \quad ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

Einstein, therefore, requires as a first restriction that (84) shall reduce to (85) by a suitable transformation of the variables and for a sufficiently small region about a given point, i.e., neglecting infinitesimals of a higher order. This amounts to Einstein's celebrated *principle of equivalence*. The further determination of the a 's depends upon the presence of material bodies and electricity. For brevity we shall consider only a special case of a gravitational field. In a system of bodies removed from all other influences, i.e., a *complete system*, the most important facts relate to the conservation of energy and momentum. In the restricted theory this is expressed by the vanishing of the energy-momentum tensor T of that theory. Einstein carries this over and takes, as a generalization of T , a symmetric tensor covering a wide class of phenomena, whose components are in contravariant form

$$(86) \quad T^{\lambda\mu} = \rho \frac{dx_\lambda}{ds} \frac{dx_\mu}{ds},$$

or in the equivalent covariant form

$$(86a) \quad T_{ij} = \sum_{\lambda, \mu} \rho a_{i\lambda} a_{j\mu} \frac{dx_\lambda}{ds} \frac{dx_\mu}{ds},$$

where ρ is the density of matter and ds is given by (84). Thus Einstein requires

$$(87) \quad \operatorname{div} \{T^{ij}\} = 0 \quad \text{or} \quad \operatorname{div} \{T_{ij}\} = 0,$$

either one of these equations having the other as a consequence.*

* A few words of explanation may be welcome to some readers. Let us recall the equations of motion of an elastic body (e.g., viscous fluid). At the point $x = (x_1, x_2, x_3)$ let the components of the stress p_i on a plane perpendicular to the x_i axis be denoted by p_{ij} , $j = 1, 2, 3$. Let u_i be the components of the velocity of the element of mass dm of density ρ at the point x . Then, when external forces are neglected,

$$(a) \quad \sum_j \frac{\partial p_{ij}}{\partial x_j} + \rho \frac{du_i}{dt} = 0, \quad i = 1, 2, 3.$$

Here $du_i/dt = \partial u_i/\partial t + \sum_j u_j \partial u_i/\partial x_j = Du_i/Dt$ in English works. To these we add the equation of continuity

$$(b) \quad \frac{\partial \rho}{\partial t} + \sum_j \frac{\partial (\rho u_j)}{\partial x_j} = 0.$$

The four equations (a), (b) determine the four unknowns ρ, u_1, u_2, u_3 .

Let us see how these equations look in the restricted theory of relativity. For simplicity we shall choose our units so that the velocity of light in vacuo $c = 1$. If we set

$$(c) \quad q_{ij} = p_{ij} + \rho u_i u_j, \quad i, j = 1, 2, 3,$$

the energy-momentum tensor T has the components

$$(d) \quad \begin{array}{cccc} q_{11} & q_{12} & q_{13} & \rho u_1, \\ q_{21} & q_{22} & q_{23} & \rho u_2, \\ q_{31} & q_{32} & q_{33} & \rho u_3, \\ \rho u_1 & \rho u_2 & \rho u_3 & \rho. \end{array}$$

This is a *physical* requirement. The question now is, how does this gravitating matter affect the metric of the surrounding space? In the older mechanics the gravitational field is determined by Poisson's equation,

$$(88) \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho.$$

The left side of (88) is a linear function of the second derivatives of the potential function V , and this function, as the right side of (88) shows, is proportional to the amount of matter per unit volume. Now, in the restricted theory of relativity, mass and energy are proportional. This leads one to generalize by assuming that the effect of matter on the metric of space is obtained by setting the energy momentum tensor proportional to a space tensor of order 2 and, by analogy to (88), we shall take one which is linear in the second derivatives of a_{ij} . The most natural tensor of this kind to take would be $\{G_{ij}\}$ defined in (67), but, unfortunately, the divergence of this tensor is not 0 and hence a relation of the type $G_{ij} = \alpha T_{ij}$ would contradict (87). From G_{ij} we can however deduce a tensor whose divergence is zero by adding the term $-\frac{1}{2}a_{ij}G$, where G

The motion of the body is now determined by the equation

$$(e) \quad \text{div } T = 0,$$

where, in general, the divergence of a tensor A whose components are $A^{\lambda\mu}$, $\lambda, \mu = 1, 2, 3, 4$ is a vector whose components are

$$(f) \quad \frac{\partial A^{\lambda_1}}{\partial x_1} + \frac{\partial A^{\lambda_2}}{\partial x_2} + \frac{\partial A^{\lambda_3}}{\partial x_3} + \frac{\partial A^{\lambda_4}}{\partial x_4}.$$

Thus the equation (e) is equivalent to four equations. For $\lambda = i = 1, 2, 3$, $x_4 = t$ it gives $\Sigma_{j=1}^3 \partial p_{ij} / \partial x_j + \partial(\rho u_i) / \partial t$ or, using (c),

$$(g) \quad \Sigma_j \frac{\partial p_{ij}}{\partial x_j} + \Sigma_j \frac{\partial(\rho u_i u_j)}{\partial x_j} + \frac{\partial(\rho u_i)}{\partial t} = 0.$$

For $\lambda = 4$, (e) gives

$$(h) \quad \Sigma_j \frac{\partial(\rho u_j)}{\partial x_j} + \frac{\partial \rho}{\partial t} = 0.$$

We note that equation (h) has the same form as (b). To reduce (g) to the form (a) we observe

$$\Sigma_j \frac{\partial(\rho u_i u_j)}{\partial x_j} + \frac{\partial(\rho u_i)}{\partial t} = u_i \Sigma \frac{\partial(\rho u_j)}{\partial x_j} + \rho \Sigma u_j \frac{\partial u_i}{\partial x_j} + \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t}$$

or, using (h),

$$= \rho \left\{ \frac{\partial u_i}{\partial t} + \Sigma_j u_j \frac{\partial u_i}{\partial x_j} \right\} = \rho \frac{du_i}{dt}.$$

Hence $\Sigma_j \partial p_{ij} / \partial x_j + \rho du_i / dt = 0$, which has the same form as (a).

To show that (e) is a special case of Einstein's equation (87) or $\text{div } \{T^{ij}\} = 0$, we recall that, when the coefficients a_{ij} in (84) are constants, as they are in the restricted theory of relativity, covariant differentiation and ordinary differentiation are the same. Then by (83) the components of $\text{div } \{T^{ij}\}$ are

$$\frac{\partial T^{\lambda_1}}{\partial x_1} + \frac{\partial T^{\lambda_2}}{\partial x_2} + \frac{\partial T^{\lambda_3}}{\partial x_3} + \frac{\partial T^{\lambda_4}}{\partial x_4}, \quad \lambda = 1, 2, 3, 4,$$

which are identical with (f).

is the curvature tensor defined by (75). The required relation is thus obtained by setting the space tensor $G_{ij} - \frac{1}{2}a_{ij}G$ proportional to the energy momentum tensor T_{ij} . We have therefore as the 10 equations to determine the 10 unknown a_{ij}

$$(89) \quad G_{ij} - \frac{1}{2}a_{ij}G = -\kappa T_{ij}, \quad i, j = 1, 2, 3, 4.$$

We can give these equations another form which is useful. Multiplying (89) by a^{ij} and summing we get

$$(90) \quad \sum_{i,j} a^{ij}G_{ij} - \frac{1}{2}G \sum a_{ij}a^{ij} = -\kappa \sum a^{ij}T_{ij}.$$

Now by (74) $\sum_{i,j} a_{ij}a^{ij} = 4$, since $n = 4$. Hence, if we set

$$T = \sum_{i,j} a^{ij}T_{ij},$$

the energy momentum invariant, we get $G - 2G = T$, which, set in (90), gives

$$(91) \quad G = \kappa T.$$

Putting this in (89) gives the desired relation

$$(92) \quad G_{ij} = -\kappa(T_{ij} - \frac{1}{2}a_{ij}T).$$

When there is no matter present at the point x , T and T_{ij} vanish. Thus for space outside gravitating matter the 10 coefficients a_{ij} are determined by the 10 differential equations

$$(93) \quad G_{ij} = 0,$$

which are linear in the second partial derivatives of the $a_{\lambda\mu}$. By means of these equations together with the radial symmetry of space we may show that the metric of our space produced by the gravitation of a central body, as the sun may be given the form expressed in (9). The constant κ which figures in (89) and (92) is found to have the value

$$(94) \quad \kappa = 8\pi \frac{k^2}{c^2} = 2 \cdot 10^{-27} \text{ c.g.s. units.}$$

14. Cosmological considerations. In studying the behavior of a complete system it is often a great convenience in ordinary mathematical physics to replace the boundary conditions by giving their values at infinity. This device was used by Einstein in his celebrated paper on the perihelion of Mercury (1915). He supposed the a_{ij} to take on the values at infinity (for a proper set of coördinates) given by the scheme

$$(95) \quad \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & c^2 \end{array}$$

which correspond to the metric $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ of the restricted theory of relativity. A disadvantage of this assumption lies in the fact that these values are tied down to a certain set of coördinates, they are not invariant. For this and other reasons (stability of our stellar system) Einstein was led to adopt a universe of finite magnitude, i.e., an elliptic metric. In this space there are no boundary values as there is no boundary. He supposes that matter is on the average uniformly distributed of density ρ . The stars are concentrations of this matter, whose greater density is compensated by a rarity of matter elsewhere. As the metric of his 4-way space he takes

$$(96) \quad ds^2 = c^2 dt^2 - \frac{dx_1^2 + dx_2^2 + dx_3^2}{\left[1 + \frac{k^2}{4}(x_1^2 + x_2^2 + x_3^2)\right]^2}.$$

It turns out, however, that this metric is in conflict with the equations (89) if we assume that the world matter has a velocity small in comparison with light, an assumption justified by the relatively small velocity of the stars so far as ascertained. This difficulty is easily remedied by introducing a new term in (89). For the left-hand side of (89) was chosen as the simplest covariant tensor of order 2 whose divergence was zero. Now we saw that the covariant derivatives of the fundamental tensor, viz., $a_{ij|k}$, are all zero. Hence by (82) $\text{div } a_{ij} = 0$. Thus we may add a term λa_{ij} ($\lambda = \text{constant}$) to the left side of (89) and still have a tensor whose divergence vanishes. Einstein therefore sets

$$(97) \quad G_{ij} - \lambda a_{ij} - \frac{1}{2} a_{ij} G = -\kappa T_{ij},$$

which is now in harmony with (96). As before we now find

$$(98) \quad G = 4\lambda + \kappa T,$$

which, set in (97), gives

$$(99) \quad G_{ij} - \lambda a_{ij} = -\kappa(T_{ij} - \frac{1}{2} a_{ij} T).$$

Where there is no matter, T and T_{ij} vanish and (99) becomes

$$(100) \quad G_{ij} - \lambda a_{ij} = 0,$$

which takes the place of the former equations (93).

The two universal constants κ, λ are related by

$$(101) \quad \lambda = \frac{1}{2} \kappa \rho, \quad \kappa \rho = \frac{2}{R^2},$$

where $1/R = k$ is the curvature of the x_1, x_2, x_3 space R_3 obtained by setting $t = \text{const.}$ in (96).

15. Estimation of the size of the universe. Astronomers often use as a unit of distance 1 parsec which equals the distance of a star whose parallax

is $1''$. Thus $1 \text{ parsec} = 2 \cdot 10^5 \text{ orbrads} = 2 \cdot 10^5 \cdot 150 \cdot 10^6 \text{ kilometers}$, or $1 \text{ parsec} = 3 \cdot 10^{18} \text{ cm.}$ Let us now assume with Kapteyn that the density of the cosmos is about the same as in a cube described about the sun and having a side of $10 \text{ parsecs} = 3 \cdot 10^{19} \text{ cm.}$ The volume of the cube is therefore $27 \cdot 10^{57} \text{ cm.}^3$. In such a cube Kapteyn estimates that there are about 80 suns of about the mass of ours. As the mass of our sun is about $2 \cdot 10^{33} \text{ gm.}$, the mass of these suns is $16 \cdot 10^{34} \text{ gms.}$, we have

$$\text{density} = \rho = \frac{\text{mass}}{\text{volume}} = \frac{16 \cdot 10^{34}}{27 \cdot 10^{57}} = 5,9 \cdot 10^{-24}.$$

From (101) we have $R^2 = 2/\kappa\rho$ and, as by (94) $\kappa = 2 \cdot 10^{-7}$, we have

$$R^2 = \frac{2}{2 \cdot 10^{-27}} \cdot \frac{1}{5,9 \cdot 10^{-24}} = 1,7 \cdot 10^{50},$$

therefore

$$R = 1,3 \cdot 10^{25} \text{ cm.} = 9 \cdot 10^{11} \text{ orbrads.}$$